# *A Class of Uniform Transcendental Functions.*

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In the Comptes Rendus for 1878, M. Picard has given two methods of forming a certain uniform transcendental function, viz. a function satisfying the conditions

$$F(u+2\omega_1)=F(u),$$
$$F(u+2\omega_3)=F(u)\,S(u),$$

where $S(u)$ is a doubly periodic function having $2\omega_1$ and $2\omega_3$ as periods. So far as I am aware, this function has not been considered by any one else since M. Picard first announced it. In what follows I have given another mode of forming the function, based upon the knowledge of its zeros and poles as found by M. Picard.

Let $q_1, q_2, \ldots . q_\pi$ denote the zeros and $p_1, p_2, \ldots . p_\pi$ the poles of the uniform doubly periodic function $S(u)$. We can take

$$q_1+q_2+\ldots .+q_\pi=p_1+p_2+\ldots .+p_\pi.$$

Picard shows that the zeros of $F(u)$ are given by

$$u=2m\omega_1+2(n+1)\,\omega_3+q_i,\quad u=2m\omega_1-2(n-1)\,\omega_3+p_j,$$

and its poles by

$$u=2m\omega_1+2(n+1)\,\omega_3+p_i,\quad u=2m\omega_1-2(n-1)\,\omega_3+q_j,$$

where $m$ takes all positive and negative integer values from $-\infty$ to $+\infty$ and $n \geqq 1$, and where each zero and each pole is of order of multiplicity $n$. Write these zeros and poles in the form

(zeros), $\quad u=s_q,\quad u=s'_p,$
(poles), $\quad u=s_p,\quad u=s'_q.$

It is to be noticed that $s$ becomes $s'$ by changing $n$ into $-n$. Let us first form a function having $s_q$ as zeros of orders of multiplicity equal to the corresponding value of $n$. To find the genus of the required function we need to know for what value of $\mu$ the series

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^{\mu}} \tag{1}$$

is convergent. Write $q=a+ib$, $\omega_1=\alpha_1+i\beta_1$, $\omega_2=\alpha_3+i\beta_3$; the modulus of the quantity in the denominator is

$$[(a+2\alpha_3+2m\alpha_1+2n\alpha_3)^2+(b+2\beta_3+2m\beta_1+2n\beta_3)^2]^{\frac{\mu}{2}}.$$

Employing now a known theorem of Jordan's in the same way that Picard employs it in Vol. I, p. 272, of his Traité d'Analyse, we compare the general term of this series with the general term of the series

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n}{(m^2+n^2)^{\mu}}; \tag{2}$$

that is, take the ratio of the general terms in (1) and (2), this is

$$\frac{[(a+2\alpha_3+2m\alpha_1+2n\alpha_3)^2+(b+2\beta_3+2m\beta_1+2n\beta_3)^2]^{\frac{\mu}{2}}}{(m^2+n^2)^{\mu}}.$$

This is never infinite or zero; we can choose a finite constant $k$, then, so that the terms in (1) shall be less than the terms in

$$\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n}{k^{\mu}(m^2+n^2)^{\mu}}.$$

If, then, (2) is convergent, (1) will also be convergent. Consider the double integral

$$\int_{y=1}^{\infty}\int_{x=-\infty}^{\infty}\frac{y\,dx\,dy}{(x^2+y^2)^{\mu}}.$$

Write $x=\rho\cos\theta$, $y=1+\rho\sin\theta$; the limits of the integration are now from $\rho=0$ to $\rho=\infty$, from $\theta=0$ to $\theta=\pi$. The integral is now

$$\int_0^{\infty}\int_0^{\pi}\frac{(1+\rho\sin\theta)\,\rho d\rho d\theta}{(1+2\rho\sin\theta+\rho^2)^{\mu}}<\int_0^{\infty}\int_0^{\pi}\frac{(1+\rho)\,\rho d\rho\,d\theta}{(1+\rho^2)^{\mu}}$$

Since we are dealing with large values of $\rho$, we have $\rho < \rho^2$, so the integral is less than

$$\int_0^\infty \int_0^\pi \frac{\rho \, d\rho \, d\theta}{(1+\rho^2)^{\mu-1}}.$$

Writing $\rho^2 = t$, this is

$$\frac{1}{2} \int_0^\infty \int_0^\pi \frac{dt \, d\theta}{(1+t)^{\mu-1}}.$$

That this may be finite for infinitely large values of $t$ we must have

$$\mu - 1 > 1 \text{ or } \mu > 2;$$

the series

$$\sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^3}$$

will then be convergent, and the function having the quantities

$$q + 2m\omega_1 + 2(n+1)\omega_3$$

as zeros of order $n$ is of genus 2. The function that we are in search of will then be of the form

$$H(u) = e^{G(u)} \prod_{i=1}^{i=\pi} \left[ \frac{\prod_{n=1}^{\infty} \prod_{m=-\infty}^{\infty} \left\{ \left(1 - \frac{u}{s_q}\right) e^{\frac{u}{s_{q_i}} + \frac{1}{2}\left(\frac{u}{s_{q_i}}\right)^2} \right\}^n \left\{ \left(1 - \frac{u}{s'_{p_i}}\right) e^{\frac{u}{s'_{p_i}} + \frac{1}{2}\left(\frac{u}{s'_{p_i}}\right)^2} \right\}^n}{\prod_{n=1}^{\infty} \prod_{m=-\infty}^{\infty} \left\{ \left(1 - \frac{u}{s_{p_i}}\right) e^{\frac{u}{s_{p_i}} + \frac{1}{2}\left(\frac{u}{s_{p_i}}\right)^2} \right\}^n \left\{ \left(1 - \frac{u}{s'_{q_i}}\right) e^{\frac{u}{s'_{q_i}} + \frac{1}{2}\left(\frac{u}{s'_{q_i}}\right)^2} \right\}^n} \right],$$

where $G(u)$ is a holomorphic function of $u$. To study this it will be sufficient to take a single one of the factors in [ ]. Say

$$F(u) = \frac{\prod_{n=1}^{\infty} \prod_{m=-\infty}^{\infty} \left\{ \left(1 - \frac{u}{s_q}\right) e^{\frac{u}{s_q} + \frac{1}{2}\frac{u^2}{s_q^2}} \right\}^n \left\{ \left(1 - \frac{u}{s'_p}\right) e^{\frac{u}{s'_p} + \frac{1}{2}\frac{u^2}{s_p'^2}} \right\}^n}{\Pi \, \Pi \left\{ \left(1 - \frac{u}{s_p}\right) e^{\frac{u}{s_p} + \frac{1}{2}\frac{u^2}{s_p^2}} \right\}^n \left\{ \left(1 - \frac{u}{s'_q}\right) e^{\frac{u}{s'_q} + \frac{1}{2}\frac{u^2}{s_q'^2}} \right\}^n}.$$

Take the logarithmic derivative of this with respect to $u$ and write $\dfrac{F'(u)}{F(u)} = \Omega(u)$. We have then

$$\Omega(u) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \left[ \left\{ \frac{n}{u - s_q} + \frac{n}{s_q} + \frac{nu}{s_q^2} \right\} - \left\{ \frac{n}{u - s_p} + \frac{n}{s_p} + \frac{nu}{s_p^2} \right\} \right].$$

The summation with respect to $n$ now going from $-\infty$ to $+\infty$, and clearly the value $n=0$ which was at first excluded need no longer be excluded, not even in the value given for $H(u)$.

Differentiating again we have

$$\Omega'(u) = -\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left[\left\{\frac{n}{(u-s_q)^2}-\frac{n}{s_q^2}\right\}-\left\{\frac{n}{(u-s_p)^2}-\frac{n}{s_p^2}\right\}\right],$$

$$\Omega''(u) = +2\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left[\frac{n}{(u-s_q)^3}-\frac{n}{(u-s_p)^3}\right].$$

Writing this last out more fully, it is

$$\Omega''(u)=2\sum\sum\left\{\frac{n}{[u-q-2m\omega_1-2(n+1)\,\omega_3]^3}-\frac{n}{[u-p-2m\omega_1-2(n+1)\omega_3]^3}\right\}.$$

We have manifestly $\Omega''(u+2\omega_1)=\Omega''(u)$. Adding $2\omega_3$ to $u$ we get after a slight arrangement of the terms

$$\Omega''(u+2\omega_3) = 2\sum\sum\left\{\frac{n-1}{[u-q-2m\omega_1-2n\omega_3]}-\frac{n-1}{[u-p-2m\omega_1-2n\omega_3]^3}\right\}$$
$$+2\sum\sum\left\{\frac{1}{[u-q-2m\omega_1-2n\omega_3]^3}-\frac{1}{[u-p-2m\omega_1-2n\omega_3]^3}\right\}.$$

This arrangement of the terms is legitimate, as each of the double series written here is absolutely convergent. This is now

$$\Omega''(u+2\omega_3)=\Omega''(u)-\wp'(u-q)+\wp'(u-p).$$

In like manner we can see that the series for $\Omega'(u)$ gives

$$\Omega'(u+2\omega_1)=\Omega'(u),$$
$$\Omega'(u+2\omega_3)=\Omega'(u)-\wp(u-q)+\wp(u-p).$$

Or these relations for $\Omega'$ could be got from those for $\Omega''$ by integration, viz. integrating

$$\Omega''(u+2\omega_1)=\Omega''(u),$$

we have

$$\Omega'(u+2\omega_1)=\Omega'(u)+c.$$

Now $\Omega'(0)=0$, so $\Omega'(2\omega_1)=c$; but if we make $u=2\omega_1$ in the above series, we see, remembering that the series is absolutely convergent, that the terms cancel in pairs, so that $\Omega'(2\omega_1)=0$.

Suppose we make $u=2\omega_3$ in $\Omega'(u)$. We have

$$\Omega'(2\omega_3)=-\sum\sum\left[\left\{\frac{n}{[q+2m\omega_1+2n\omega_3]^2}-\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^2}\right\}-\left\{p\right\}\right],$$

$\{p\}$ being the same function of $p$ that the term preceding it is of $q$. Grouping the terms a little differently, we see that they all cancel in pairs except the terms

$$-\sum\sum\left[\frac{1}{[q+2m\omega_1+2(n+1)\omega_3]^2}-\frac{1}{[p+2m\omega_1+2(n+1)\omega_3]^2}\right].$$

These are two series, of course, one in $q$ and the other in $p$; neither is convergent; but if we subtract the quantity

$$\frac{1}{[2m\omega_1+2(n+1)\omega_3]^2}$$

from each of the terms in [ ], we leave the value of the terms in [ ] unaltered, but the series in $q$ and in $p$ are now each convergent, and the whole sum is now equal to

$$-\wp(q)+\wp(p)=\Omega'(2\omega_3).$$

(Of course for $m=0$ and $n+1=0$ the terms of the series are simply $\frac{-1}{q^2}$ and $\frac{1}{p^2}$).
Integrate now the relation

$$\Omega''(u+2\omega_3)=\Omega''(u)-\wp'(u-q)+\wp'(u-p);$$

we have

$$\Omega'(u+2\omega_3)=\Omega'(u)-\wp(u-q)+\wp(u-p)+c.$$

Make $u=0$, then since $\Omega'(0)=0$, we have

$$\Omega'(2\omega_3)=-\wp(q)+\wp(p)+c,$$

it follows then that $c=0$, and so that

$$\Omega'(u+2\omega_3)=\Omega'(u)-\wp(u-q)+\wp(u-p).$$

In like manner we could find the result of changing $u$ into $u+2\omega_1$ and $u+2\omega_3$ in $\Omega(u)$ and $F(u)$. We shall proceed differently, however, following a method used by Taumery and Molk in their *Fonctions elliptiques*, p. 160.

From $F(u+a)$: considering only one of the factors in this, say

$$\left(1-\frac{u+a}{s_q}\right)e^{\frac{u+a}{s_q}+\frac{1}{2}\left(\frac{u+a}{s_q}\right)^2},$$

we notice that it can be put in the form of the product of the three quantities

$$\left(1-\frac{u}{s_q-a}\right)e^{\frac{u}{s_q-a}+\frac{1}{2}\left(\frac{u}{s_q-a}\right)^2},$$

$$\left(1-\frac{a}{s_q}\right)e^{\frac{a}{s_q}+\frac{1}{2}\left(\frac{a}{s_q}\right)^2},$$

$$e^{u\left[\frac{1}{a-s_q}+\frac{1}{s_q}+\frac{a}{s_q^2}\right]-\frac{u^2}{2}\left[\frac{1}{(a-s_q)^2}-\frac{1}{s_q^2}\right]}.$$

If we make a similar decomposition of all the factors and then combine the results, we find quite readily the relation

$$\left[\frac{F(u+a)}{F(a)}\right]e^{-u\Omega(a)-\frac{u^2}{2}\Omega'(a)}$$

$$=\frac{\prod\limits_{n=1}^{\infty}\prod\limits_{m=-\infty}^{\infty}\left\{\left(1-\frac{u}{s_q-a}\right)e^{\frac{u}{s_q-a}+\frac{1}{2}\frac{u^2}{(s\ -a)^2}}\right\}^n\left\{\left(1-\frac{u}{s'_p-a}\right)e^{\frac{u}{s_p-a}+\frac{1}{2}\frac{u^2}{(s'_p-a)^2}}\right\}^n}{\prod\limits_{n=1}^{\infty}\prod\limits_{m=-\infty}^{\infty}\left\{\left(1-\frac{n}{s'_q-a}\right)e^{\frac{u}{s'_q-a}+\frac{1}{2}\frac{u^2}{(s'_q-a)^2}}\right\}^n\left\{\left(1-\frac{u}{s_p-a}\right)e^{\frac{u}{s_p-a}+\frac{1}{2}\frac{u^2}{(s_p-a)^2}}\right\}^n}.$$

Differentiating logarithmically with respect to $u$, we have

$$\Omega(u+a)-\Omega(a)-u\Omega'(a)=\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left[\left\{\frac{n}{u+a-s_q}-\frac{n}{a-s_q}+\frac{nu}{(a-s_q)^2}\right\}\right.$$
$$\left.-\left\{\frac{n}{u+a-s_p}-\frac{n}{a-s_p}+\frac{nu}{(a-s_p)^2}\right\}\right].$$

Differentiating again gives

$$\Omega'(u+a)-\Omega'(a)=-\sum\sum\left[\left\{\frac{n}{(u+a-s_q)^2}-\frac{n}{(a-s_q)^2}\right\}\right.$$
$$\left.-\left\{\frac{n}{(u+a-s_p)^2}-\frac{n}{(a-s_p)^2}\right\}\right].$$

This last will give us nothing but what we have already found. In the preceding one, change $a$ into $a+2\omega_1$ and we get

$$\Omega(u+a+2\omega_1)-\Omega(a+2\omega_1)-u\Omega'(a+2\omega_1)=\Omega(u+a)-\Omega(a)-u\Omega'(a),$$

since the series on the right is manifestly unaltered by changing $a$ into $a + 2\omega_1$. Write $u - a$ for $u$ in this last; this gives

$$\Omega(u + 2\omega_1) - \Omega(u) = \Omega(a + 2\omega_1) - \Omega(a),$$

since $\Omega'(a + 2\omega_1) = \Omega'(a)$.

We have then

$$\Omega(u + 2\omega_1) - \Omega(u) = c, \text{ a constant.}$$

Making $u = 0$, and noticing that $\Omega(0) = 0$, gives

$$\Omega(2\omega_1) = c.$$

In the equation giving $\Omega$ write $u = 2\omega_1$ and we have

$$\Omega(2\omega_1) = \sum\sum\left\{\frac{-n}{q + 2(m-1)\omega_1 + 2(n+1)\omega_3} + \frac{n}{q + 2m\omega_1 + 2(n+1)\omega_3} + \frac{2\omega_1 n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2}\right\} - \left\{p\right\}.$$

The first and second terms in each $\{\ \}$ will cancel out when the whole sum is considered, and the third terms give, by adding and subtracting

$$\frac{2n\omega_1}{[2m\omega_1 + 2(n+1)\omega_3]^2},$$

the convergent series

$$\Omega(2\omega_1) = 2\omega_1\left[\sum\sum\left\{\frac{n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2} - \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^2}\right\} - \left\{p\right\}\right],$$

say $\Omega(2\omega_1) = \lambda_1$. Make $u = 2\omega_3$ and we have

$$\Omega(2\omega_3),\ = \lambda_3,\ = \sum\sum\left\{\frac{-n}{q + 2m\omega_1 + 2n\omega_3} + \frac{n}{q + 2m\omega_1 + 2(n+1)\omega_3} + \frac{2\omega_3 n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2}\right\} - \left\{p\right\}.$$

Here it will be desirable to introduce all the zeros and poles $q$ and $p$ of the pro-

posed doubly periodic function. Let $\Omega_i$ denote the $\Omega$ corresponding to $(p_i, q_i)$; then

$$\lambda_1 = \Omega(2\omega_1) = \sum \Omega_i(2\omega_1) = \sum \lambda_1^{(i)}$$

$$= 2\omega_1 \sum_{i=1}^{i=\pi} \Big[ \sum \sum \Big\{ \frac{n}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]} - \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^2} \Big\} - \Big\{ p \Big\} \Big] = 2\omega_1 R,$$

$$\lambda_3 = \Omega(2\omega_3) = \sum \Omega_i(2\omega_3) = \sum \lambda_3^{(i)}$$

$$= \sum_{i=1}^{i=\pi} \Big[ - \sum \sum \Big\{ \frac{1}{q_i + 2m\omega_1 + 2(n+1)\omega_3} + \frac{1}{2m\omega_1 + 2(n+1)\omega_3} + \frac{q_i}{[2m\omega_1 + 2(n+1)\omega_3]^2} \Big\} - \Big\{ p \Big\} \Big]$$

$$+ 2\omega_3 \sum_{i=1}^{i=\pi} \Big[ \sum \sum \Big\{ \frac{n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2} - \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^2} \Big\} - \Big\{ p \Big\} \Big].$$

In the first line of this we have introduced the terms

$$\frac{1}{2m\omega_1 + 2(n+1)\omega_3} + \frac{q_i}{[2m\omega_1 + 2(n+1)\omega_2]^2} - \frac{1}{2m\omega_1 + 2(n+1)\omega_3} - \frac{p_i}{[2m\omega_1 + 2(n+1)\omega_3]^2},$$

but since we assume $\Sigma p_i = \Sigma q_i$, no change is produced in the value of the series. We have now

$$\lambda_3 = \Sigma[-\zeta(q_i) + \zeta(p_i)] + 2\omega_3 R,$$

and so

$$\lambda_3\omega_1 - \lambda_1\omega_3 = \omega_1 \Sigma [\zeta(p_i) - \zeta(q_i)].$$

Let $\Omega$ denote the sum $\sum_{i=1}^{i=\pi} \Omega i$, we have now

$$\Omega(u + 2\omega_1) = \Omega(u) + \lambda_1,$$
$$\Omega(u + 2\omega_3) = \Omega(u) + \Sigma\zeta(u - q_i) - \Sigma\zeta(u - p_i) + \lambda_3.$$

Finally, letting $F$ denote the product $\prod\limits_{i=1}^{i=\pi} F_i$, we get

$$F(u+2\omega_1) = F(u)\, e^{\lambda_1 u + \kappa_1},$$

$$F(u+2\omega_3) = F(u)\, \frac{\sigma(u-q_1)\ldots\ldots\sigma(u-q_\pi)}{\sigma(u-p_1)\ldots\ldots\sigma(u-p_\pi)}\, e^{\lambda_3 u + \kappa_3},$$

where

$$\varkappa_1 = \log F(2\omega_1),$$

$$\varkappa_3 = \log F(2\omega_3) + \Sigma \log \sigma p_i - \Sigma \log \sigma q_i .$$

Form now the quadratic function

$$g(u) = Au^2 + Bu,$$

and determine $A$ and $B$ so that

$$4A\omega_1 = -\lambda_1, \quad 4A\omega_1^2 + 2B\omega_1 = -\varkappa_1,$$

that is

$$A = -\frac{\lambda_1}{4\omega_1}, \quad B = \frac{\lambda_1\omega_1 - \varkappa_1}{2\omega_1}.$$

Writing now

$$G(u) = e^{g(u)} F(u)$$

and we have

$$G(u+2\omega_1) = G(u),$$

$$G(u+2\omega_3) = G(u)\, \frac{\sigma(u-q_1)\ldots\ldots\sigma(u-q_\pi)}{\sigma(u-p_1)\ldots\ldots\sigma(u-p_\pi)}\, e^{\alpha u + \beta},$$

where

$$\alpha = \frac{1}{\omega_1}\left[\lambda_3\omega_1 - \lambda_1\omega_3\right] = \Sigma\left[\zeta(p_i) - \zeta(q_i)\right],$$

$$\beta = \frac{\lambda_1\omega_3}{\omega_1}(\omega_1 - \omega_3) + \frac{\varkappa_3\omega_1 - \varkappa_1\omega_3}{\omega_1}.$$

I cannot see any error in the above considerations on the convergence of the series (1), and consequently in the determination of the genus of the function $F(u)$ or $H(u)$; but from certain analogies with the known series used in elliptic functions, it seems possible that we should have $\mu = 4$ to secure the convergence of (1), and consequently that the genus of the function is three at least; three is a certain value. In this case we would have (using for the moment a single $q$ and a single $p$),

$$F(u) = \frac{\prod\limits_{n=1}^{\infty} \prod\limits_{m=-\infty}^{\infty} \left\{\left(1 - \frac{u}{s_q}\right) e^{\frac{u}{s_q} + \frac{1}{2}\frac{u^2}{s_q^2} + \frac{1}{3}\frac{u^3}{s_q^3}}\right\}_n^n \left\{s_p'\right\}^n}{\prod \prod \left\{\left(1 - \frac{u}{s_p}\right) e^{\frac{u}{s_p} + \frac{1}{2}\frac{u^2}{s_p^2} + \frac{1}{3}\frac{u^3}{s_p^3}}\right\}^n \left\{s_q'\right\}^n},$$

and

$$\Omega\ (u) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left[\left\{\frac{n}{u-s_q}+\frac{n}{s_q}+\frac{nu}{s_q^2}+\frac{nu^2}{s_q^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega'\ (u) = -\sum\sum\left[\left\{\frac{n}{(u-s_q)^2}-\frac{n}{s_q^2}-\frac{2nu}{s_q^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega''(u) = 2\sum\sum\left[\left\{\frac{n}{(u-s_q)^3}+\frac{n}{s_q^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega'''(u) = -6\sum\sum\left[\frac{n}{(u-s_q)^4}-\frac{n}{(u-s_p)^4}\right].$$

Again changing $u$ into $u+a$ and decomposing the general factor as above, we obtain

$$\frac{F(u+a)}{F(a)}e^{-u\Omega(a)-\frac{u^2}{2}\Omega'(a)-\frac{u^3}{6}\Omega''(a)}$$

$$= \frac{\prod\limits_{n=1}^{\infty}\prod\limits_{m=-\infty}^{\infty}\left\{\left(1-\frac{u}{s_q-a}\right)e^{\frac{u}{s_q-a}+\frac{1}{2}\frac{u^2}{(s_q-a)^2}+\frac{1}{3}\frac{u^3}{(s_q-a)^3}}\right\}^n\left\{s_p'\right\}^n}{\prod\prod\left\{\left(1-\frac{u}{s_p-a}\right)e^{\frac{u}{s_p-a}+\frac{1}{2}\frac{u^2}{(s_p-a)^2}+\frac{1}{3}\frac{u^3}{(s_p-a)^3}}\right\}^n\left\{s_q'\right\}^n}.$$

Differentiating logarithmically we get

$$\Omega(u+a)-\Omega(a)-u\Omega'(a)-\frac{u^2}{2}\Omega''(a)$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left[\left\{\frac{n}{u+a-s_q}-\frac{n}{a-s_q}+\frac{nu}{(a-s_q)^2}-\frac{nu^2}{(a-s_q)^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega'(u+a)-\Omega'(a)-u\Omega''(a)$$

$$= -\sum\sum\left[\left\{\frac{n}{(u+a-s_q)^2}-\frac{n}{(a-s_q)^2}+\frac{2nu}{(a-s_q)^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega''(u+a)-\Omega''(a) = 2\sum\sum\left[\left\{\frac{n}{(u+a-s_q)^3}-\frac{n}{(a-s_q)^3}\right\}-\left\{s_p\right\}\right],$$

$$\Omega'''(u+a) = -6\sum\sum\left[\frac{n}{(u+a-s_q)^4}-\frac{n}{(u+a-s_p)^4}\right].$$

The effect of adding $2\omega_1$ or $2\omega_3$ to the argument of $\Omega$, $\Omega'$ .... can be found from these by changing $a$ into $a+2\omega_1$ and $a+2\omega_3$, or it can be found from the preceding equations giving $\Omega$, $\Omega'$, $\Omega''$, $\Omega'''$ by integration; the question comes to the same thing in both cases, viz. the determination of the values of certain constants. We find at once that

$$\Omega'''(u+2\omega_1) = \Omega'''(u),$$
$$\Omega'''(u+2\omega_1) = \Omega'''(u)-\wp''(u-q)+\wp''(u-p).$$

Integrating these gives first

$$\Omega''(u+2\omega_1)=\Omega''(u)+c;$$

and since, as is readily seen, $\Omega''(2\omega_1)=0$, we have $c=0$. Again, in $\Omega''(u)$ write $u=2\omega_3$; this gives

$$\Omega''(2\omega_3)=+2\sum\sum\Big[\frac{-n}{(q+2m\omega_1+2n\omega_3)^3}+\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^3}\Big\}-\Big\{p\Big\}\Big].$$

After some obvious reductions this gives

$$\Omega''(2\omega_3)=\wp'(q)-\wp'(p).$$

Integrating the equation giving $\Omega'''(u+2\omega_1)$, making $u=0$ and using this last result, we find

$$\Omega''(u+2\omega_3)=\Omega''(u)-\wp'(u-q)+\wp'(u-p).$$

Again compute $\Omega'(2\omega_1)$ and $\Omega'(2\omega_3)$. We have first

$$\Omega'(2\omega_1)=-\sum\sum\Big[\Big\{\frac{n}{[q+2(m-1)\omega_1+2(n+1)\omega_3]^2}$$
$$-\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^2}-\frac{4\omega_1 n}{[q+2m\omega_1+2(n+1)\omega_3]^3}\Big\}-\Big\{p\Big\}\Big].$$

The first and second terms in the whole sum will cancel; we can imagine terms

$$+\frac{4\omega_1 n}{[2m\omega_1+2(n+1)\omega_1]^3}-\frac{4\omega_1 n}{[2m\omega_1+2(n+1)\omega_3]^3}$$

introduced so as to make the series of third terms in $q$ and in $p$ separately convergent; then

$$\Omega'(2\omega_1)=4\omega_1\sum\sum\Big[\Big\{\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^3}$$
$$+\frac{n}{[2m\omega_1+2(n+1)\omega_3]^3}\Big\}-\Big\{p\Big\}\Big],$$

say $$\lambda_1=\Omega'(2\omega_1)=4\omega_1 R.$$

Next form $\Omega'(2\omega_3)$ this is

$$\lambda_3=\Omega'(2\omega_3)=-\sum\sum\Big[\Big\{\frac{n}{[q+2m\omega_1+2n\omega_3]^2}-\frac{n}{[q+2m\omega_1+2(n+1)\omega_3]^2}$$
$$-\frac{4\omega_3 n}{[q+2m\omega_1+2(n+1)\omega_3]^3}\Big\}-\Big\{p\Big\}\Big].$$

Here again it is necessary to introduce the remaining $q$'s and $p$'s. Let as before $\Omega = \sum_{i=1}^{i=\pi} \Omega_i'$, and so for the other symbols. We find then readily

$$\lambda_1 = 4\omega_1 \sum_{i=1}^{i=\pi} \left[ \sum\sum \left\{ \frac{n}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]^3} + \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^3} \right\} - \{p\} \right]$$

$$= 4\omega_1 R,$$

$$\lambda_3 = -\sum_{i=1}^{i=\pi} \left[ \sum\sum \left\{ \frac{1}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2} - \frac{1}{[2m\omega_1 + 2(n+1)\omega_3]^2} \right\} - \{p\} \right]$$

$$+ 4\omega_3 \sum_{i=1}^{i=\pi} \left[ \sum\sum \left\{ \frac{n}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]^3} - \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^3} \right\} - \{p\} \right],$$

$$\sum [-\wp(q_i) + \wp(p_i)] + 4\omega_3 R.$$

The terms which have been introduced cancel each other. Now we see that

$$\lambda_1\omega_3 - \lambda_3\omega_1 = \omega_1 \sum_{i=1}^{i=\pi} [\wp(q_i) - \wp(p_i)].$$

Again write $\delta_1 = \Omega(2\omega_1) = \Sigma\Omega_i(2\omega_1)$, $\delta_3 = \Omega(2\omega_3) = \Sigma\Omega_i(2\omega_3)$. We have first

$$\delta_1 = \sum_{i=1}^{i=\pi} \left[ \sum\sum \frac{-n}{q_i + 2(m-1)\omega_1 + 2(n+1)\omega_3} + \frac{n}{q_i + 2m\omega_1 + 2(n+1)\omega_3} \right.$$
$$\left. + \frac{2n\omega_1}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]^2} + \frac{4\omega_1^2 n}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]^3} \right\} - \{p\} \Bigg]$$

$$= \sum_{i=1}^{i=\pi} \left[ 2\omega_1 \sum\sum \left\{ \frac{n}{[q_i + 2m\omega_1 + 2(n+1)\omega_3]} - \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^2} \right. \right.$$
$$\left. - \frac{2nq_i}{[2m\omega_1 + 2(n+1)\omega_3]^3} \right\} - \{p\}$$
$$\left. + 4\omega_1^2 \sum\sum \left\{ \frac{n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^3} + \frac{n}{[2m\omega_1 + 2(n+1)\omega_3]^3} \right\} - \{p\} \right].$$

The terms introduced destroy each other either identically or because $\Sigma p_i = \Sigma q_i$. Write this in the form

$$\delta_1 = 2\omega_1 S + 4\omega_1^2 T.$$

The $T$ here is the $R$ above, so

$$\delta_1 = 2\omega_1 S + 4\omega_1^2 R.$$

Again,

$$\delta_3 = \sum_i \Big[ \sum \frac{-n}{q_i + 2m\omega_1 + 2n\omega_3} + \frac{n}{q_i + 2m\omega_1 + 2(n+1)\omega_3}$$
$$+ \frac{2n\omega_3}{[q + 2m\omega_1 + 2(n+1)\omega_3]^2} + \frac{4\omega_3^2 n}{[q + 2m\omega_1 + 2(n+1)\omega_3]^3}\Big\} - \Big\{p\Big\}\Big].$$

After some easy reductions we get

$$\delta_3 = \sum_{i=1}^{i=\pi} [\zeta(p_i) - \zeta(q_i)] + 2\omega_3 S + 4\omega_3^2 R,$$

and so

$$\delta_3\omega_1 - \delta_1\omega_3 = \omega_1 \Sigma [\zeta(p_i) - \zeta(q_i)] + 4\omega_1\omega_3 R(\omega_3 - \omega_1).$$

We have now

$$\Omega'(u + 2\omega_1) = \Omega'(u) + \lambda_1,$$
$$\Omega'(u + 2\omega_3) = \Omega'(u) - \Sigma\wp(u - q_i) + \Sigma\wp(u - p_i),$$

and

$$\Omega(u + 2\omega_1) = \Omega(u) + \lambda_1 u + \delta_1,$$
$$\Omega(u + 2\omega_3) = \Omega(u) + \lambda_3 u + \delta_3$$
$$+ \Sigma[\zeta(u - q_i) - \zeta(u - p_i)].$$

Finally,

$$F(u + 2\omega_1) = F(u)\, e^{\lambda_1 \frac{u^2}{2} + \delta_1 u + \kappa_1},$$
$$F(u + 2\omega_3) = F(u) \frac{\sigma(u - q_1)\sigma(u - q_2) \ldots \sigma(u - q_\pi)}{\sigma(u - p_1)\sigma(u - p_2) \ldots \sigma(u - p_\pi)} e^{\lambda_3 \frac{u^2}{2} + \delta_3 u + \kappa_3}.$$

Choose now the cubic

$$g(u) = Au^3 + Bu^2 + Cu,$$

so that

$$6A\omega_1 u^2 + (12A\omega_1^2 + 4B\omega_1)u + 8A\omega_1^3 + 4B\omega_1^2 + 2C\omega_1 = -\lambda_1 \frac{u^2}{2} - \delta_1 u - \varkappa_1;$$

this requires

$$A = -\frac{\lambda_1}{12\omega_1}, \quad B = \frac{\lambda_1\omega_1 - \delta_1}{4\omega_1}, \quad C = -\frac{1}{6\omega_1}[\lambda_1\omega_1^2 - 3\delta_1\omega_1 + \varkappa_1].$$

Writing again

$$e^{g(u)} F(u) = G(u),$$

and it is seen that $G(u)$ satisfies the equations

$$G(u + 2\omega_1) = G(u),$$
$$G(u + 2\omega_3) = G(u) \frac{\sigma(u - q_1)\sigma(u - q_2) \ldots \sigma(u - q_\pi)}{\sigma(u - p_1)\sigma(u - p_2) \ldots \sigma(u - p_\pi)} e^{\alpha u^2 + \beta u + \gamma},$$

where

$$\alpha = \frac{1}{2\omega_1}(\lambda_3\omega_1 - \lambda_1\omega_3) = \frac{1}{2}\sum_{i=1}^{i=\pi}[\wp(q_i) - \wp(p_i)],$$

$$\beta = \frac{\lambda_1\omega_3}{\omega_1}(\omega_1 - \omega_3) + \frac{\delta_3\omega_1 - \delta_1\omega_3}{\omega_1}$$

$$= \frac{\lambda_1\omega_3}{\omega_1}(\omega_1 - \omega_3) + \Sigma\,[\zeta(p_i) - \zeta(q_i)] + 4\omega_3(\omega_3 - \omega_1)R,$$

but, as seen above, $\lambda_1 = 4\omega_1 R$, so we get finally

$$\alpha = \frac{1}{2}\sum_{i=1}^{i=\pi}[\wp(q_i) - \wp(p_i)],$$

$$\beta = \sum_{i=1}^{i=\pi}[\zeta(q_i) - \zeta(p_i)],$$

$$\gamma = -\frac{\lambda_1\omega_3}{3\omega_1}[2\omega_3^2 + \omega_1^2 - 3\omega_1\omega_3] + \frac{\delta_1\omega_3}{\omega_1}(\omega_1 - \omega_3) + \frac{\varkappa_3\omega_1 - \varkappa_1\omega_3}{\omega_1}.$$

The function $G(u)$ in this case is associated with a doubly periodic function of the third kind; in the preceding case, where the genus of the function was taken as two, the corresponding doubly periodic function is of the second kind.